DTFT

1. What is the DTFT of the following finite length sequences? You can use Matlab commands if you write them out explicitly.

   a. $x_1[n] = [\underset{\uparrow}{3} \quad 2 \quad 1]$ at $\omega = \pi/2$ rads/sample

   $$X_1(e^{j\omega}) = \sum_{n=-\infty}^{\infty} x[n]\, e^{-j\omega n}$$
   $$= 3 + 2e^{-j\omega} + e^{-j2\omega}$$
   $$X_1(e^{j\pi/2}) = 3 + 2e^{-j\frac{\pi}{2}} + e^{-j2\frac{\pi}{2}}$$
   $$= 3 - j2 + (-1) = \boxed{2-j2}$$

   or freqz([3 2 1],1,[0 pi/2])

   b. $x_2[n] = [3 \quad \underset{\uparrow}{2} \quad 3]$ for all ω

   $$X(e^{j\omega}) = 3e^{j\omega} + 2 + 3e^{-j\omega}$$
   $$= 2 + 3\cdot 2\,\frac{(e^{j\omega}+e^{-j\omega})}{2}$$
   $$= \boxed{2 + 6\cos(\omega)}$$

   c. $x_3[n] = [3 \quad 2 \quad \underset{\uparrow}{1}]$ at $\omega = 0$ rads/sample

   $$X(e^{j\cancel{\omega}0}) = 3e^{j2\cdot 0} + 2e^{j0} + 1$$
   $$= 3 + 2 + 1 = \boxed{6}$$

2. $x_4[n] = [k \quad -k \quad k \quad -k]$

   $$x[n] = [k \ -k \ k \ -k] = k\,[\underset{\uparrow}{1} \ -1 \ 1 \ -1]$$
   $$X(e^{j\omega}) = k(1 - e^{-j\omega} + e^{-j2\omega} - e^{-j\frac{3\pi}{2}})$$

   a. Find $X(e^{j0})$

   $$X(e^{j0}) = k(1 - 1 + 1 - 1) = \boxed{0}$$

   b. Find $X(e^{j\pi/2})$

   $$X(e^{j\pi/2}) = k\left(1 - e^{-j\pi/2} + e^{-j\pi} - e^{-j\frac{3\pi}{2}}\right)$$
   $$= k\,(1 - (-j) + (-1) - j)$$
   $$= \boxed{0}$$

c. Find $X(e^{j\pi})$

$$X(e^{j\pi}) = k\,(1 - e^{-j\pi} + e^{-j2\pi} - e^{-j3\pi})$$
$$= k\,(1-(-1)+1-(-1))$$
$$= \boxed{4k}$$

d. A student argues that by inspection all energy in the sequence must be at $\omega = \pi$ rads/sample since it exhibits the highest frequency oscillation possible in the digital domain.

A second student argues that the DTFT will show all the energy is concentrated at $\omega = \pi + k2\pi$ where k is any integer.

Is either one right?

$X_4[n]$ not a pure $\omega=\pi$ sinusoid; it is $[\ldots \underbrace{0\,0\,0}_{\text{would also have to be}}\ k\ -k\ k\ -k\ \underbrace{0\,0\,0}_{k\ -k\ k\ -k} \ldots]$

DFT

3. $x[n] = [1 \quad 3 \quad 2 \quad 4 \quad -1 \quad 6]$ with DFT of X[k]

Which k corresponds to the highest frequency in $x[n]$?

DFT is DTFT sampled at $\omega = \frac{k2\pi}{N}$

So to find k equivilent to $\omega = \pi$

$$\pi = \frac{k2\pi}{6}$$

Solve for k, $\boxed{k=3}$

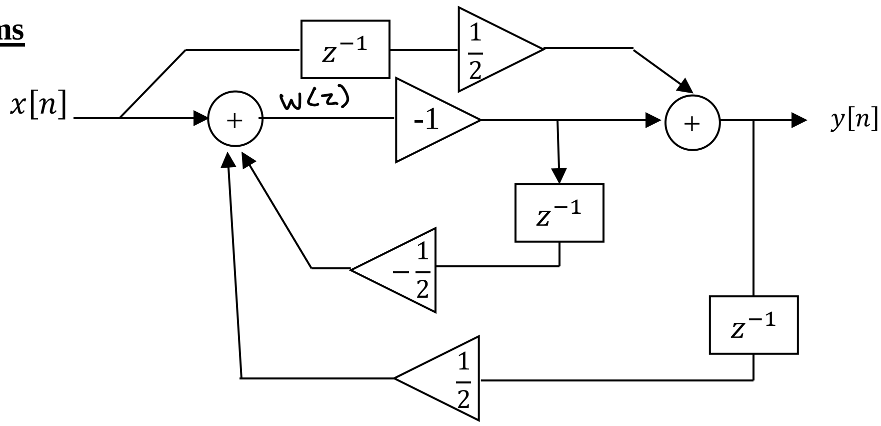
4. Find DFT of $X_4[n] = [k \quad -k \quad k \quad -k]$.

$$X[k] = \sum_{n=0}^{N-1} x[n]\, e^{-j\frac{2\pi k n}{N}}$$
$$X[0] = \sum_{n=0}^{N-1} x[n] = 0$$
$$X[1] = \sum_{n=0}^{N-1} x[n]\, e^{-j\frac{\pi}{2}n} = k\,(1+(-j)+(-1)+j) = 0$$
$$X[2] = \sum_{n=0}^{N-1} x[n]\, e^{-j\pi n} = k\,(1-(-1)+1-(-1)) = 4k$$
$$X[3] = \sum_{n=0}^{N-1} x[n]\, e^{-j\frac{3\pi}{2}n} = k\,(1+j+(-1)+(-j)) = 0$$

So $X[k] = [\underset{\uparrow}{0} \quad 0 \quad 4k \quad 0]$

Z Transforms

5.



a. Find $H(z)$

① $Y(z) = \frac{1}{2} z^{-1} X(z) - W(z) \Rightarrow W(z) = \frac{1}{2} z^{-1} X(z) - Y(z)$

② $W(z) = X(z) + \frac{1}{2} z^{-1} W(z) + \frac{1}{2} z^{-1} Y(z)$

$W(z)[1 - \frac{1}{2} z^{-1}] = X(z) + \frac{1}{2} z^{-1} Y(z)$ sub in ①

$(\frac{1}{2} z^{-1} X - Y)(1 - \frac{1}{2} z^{-1}) = X + \frac{1}{2} z^{-1} Y$ gather X, Y terms on opposite sides

$Y(1 - \frac{1}{2} z^{-1} + \frac{1}{2} z^{-1}) = X(-1 + \frac{1}{2} z^{-1} - \frac{1}{4} z^{-2})$

$Y = X(-1 + \frac{1}{2} z^{-1} - \frac{1}{4} z^{-2})$

$$H(z) = \frac{Y(z)}{X(z)} = \boxed{\frac{-1 + \frac{1}{2} z^{-1} - \frac{1}{4} z^{-2}}{1}}$$

b. For $x[n] = \delta[n]$, find $y[n]$

$H(z) = -1 + \frac{1}{2} z^{-1} - \frac{1}{4} z^{-2}$

$\Leftrightarrow$

$h[n] = -\delta[n] + \frac{1}{2} \delta[n-1] - \frac{1}{4} \delta[n-2]$

since $h[n]$ is $y[n]$ when $x[n] = \delta[n]$ (by definition)

$$\boxed{y[n] = -\delta[n] + \frac{1}{2} \delta[n-1] - \frac{1}{4} \delta[n-2]}$$