Phasor Tutorial

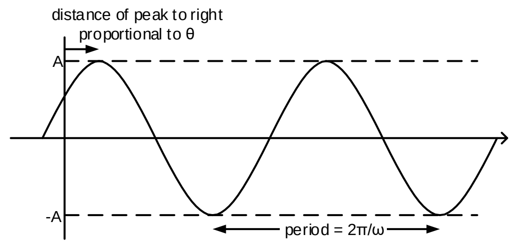
This tutorial is composed of three parts:

- a short, 2 page mathematical explanation of what phasors are and why they are important
- a single page walkthrough of an interactive, downloadable app to investigate phasors, and
- a questionnaire that will be used to determine how much (if any) interactive software tools help students learn complex theoretical subjects

Part 1: Phasors and the math behind them

What is a sinusoid?

Sinusoids, such as A cos(ω t + θ), are a backbone to engineering analysis. They occur as a natural solution to differential equations since they are similar, to within a scaling factor and angle, of their derivatives.



This family of sinusoids are fully characterized by just three numbers A, θ, and ω, as shown in the above diagram. "Characterized" means that given these three numbers we select one unique waveform from the infinite number of sinusoids that belong to the family.

A Amplitude, the maximum height, positive and negative, of the waveform

θ Phase, the horizontal displacement of the waveform.

ω Angular frequency, which is related to the waveform's period T (width of one cycle) by T = 2π/ω.

Sinusoids and Linear Systems

Any linear system that is forced by a sinusoid will respond by a sinusoid of the same frequency but with a different amplitude and phase (equivalently a different A and θ). This result follows from the fact that all derivatives of the sinusoid have the same frequency but with different values of A and θ. This result is remarkable: it means, for instance, that if we place a voltage of

$$v_{source}(t) = \cos(7t)$$

as an input to any linear circuit, that every voltage $v_{component}$ across every component in that circuit, and every current $i_{component}$ through every component in that circuit will look like

$$v_{component}(t) = A_{component} \cos(7t + \theta_{component}) \text{ and}$$
$$i_{component}(t) = A_{component} \cos(7t + \theta_{component})$$

Similarly, if we apply an external downward force to a linear car's suspension system of

$$v_{source}(t) = \cos(7t)$$

then every displacement $d_{component}$ across every suspension component and every velocity $v_{component}$ of every suspension component will look like

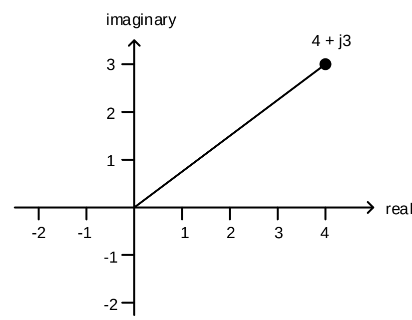
$$d_{component}(t) = A_{component} \cos(7t + \theta_{component}) \text{ and}$$
$$v_{component}(t) = A_{component} \cos(7t + \theta_{component})$$

The same is true for *every* variable that can be described as an output of *any* linear system when excited by a sinusoidal source. This means that if you model *anything* (e.g. a robotic arm, thermostat, international trade war, aircraft flight trajectory, autonomous networked cars driving on a highway, predator/prey relationship, stock market price, *anything*) using linear, constant coefficient differential equations, then a sinusoid in gives a sinusoid out. It is one of the most powerful results from linear theory.

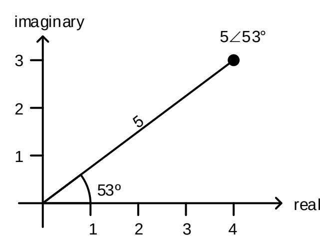
Therefore, the only interesting parts of the response of a linear system are the amplitude A and phase θ, since the frequency ω will be the same as whatever is applied. Said again: once you know the applied excitation force (or pressure, velocity, voltage, price change, whatever) is a cosine of frequency ω, without doing any mathematics you know the form of all the characteristics in the system is $A \cos(\omega t + \theta)$, where ω is the same as the excitation force. You only need to calculate A and θ.

Imaginary numbers

The square root of -1 is defined to be the imaginary number j (mathematicians call this i; engineers call it j). A complex number has both a real part and an imaginary part, like 4 + j3. This method of describing a complex number is called *rectangular form*. Another way is to think of it as a point on a plane, in which the horizontal axis is the real part of the number and the vertical axis is the imaginary part of the number, as shown to the right. This is called plotting the number on the *complex plane*.



There’s a third way to describe the complex number: *polar form*. Look at the complex plane to the right. Instead of describing the number as 4 units of real part and 3 units of imaginary part, it could be described as a point about 53° up from the horizontal axis and 5 units away from the origin.



In summary, the three ways of describing an imaginary number are

- as a point on the complex plane
- in rectangular form, like 4 + j3
- in polar form, like 5∠53°

Can we represent a sinusoid as a complex number? Phasors!

It seems impossible to represent a sinusoid by a single complex number since it takes an infinite number of points to draw a sinusoidal line. But, for a given known frequency, a sinusoid is described by just two variables (amplitude and phase angle), and a complex number in polar form is also described by just two numbers (magnitude and angle). A sinusoid of form $A \cos(\omega t + \theta)$ can therefore be described by the complex number $A\angle\theta$. This is called a **phasor**. As an example, the sinusoid $5 \cos(\omega t + 53°)$ can be represented by the phasor $5\angle 53°$.

The next section will help you develop an intuitive sense of what sinusoid is represented by a given phasor.

Part 2: The Phasor Demonstration Program

1. **Get the app running.** Point your browser to www.jimsquire.com. Go to Research and then about halfway down the page to Phasors. Download one of the two application programs as directed by your instructor by right-clicking it, choosing "Save target as…" and then saving it to the desktop). Run the program by double-clicking the program (called "Phasor" followed by a letter and digit, like "PhasorC1.exe" or "PhasorD2.exe") on your desktop.

2. **Become familiar with the phasor panel.** The complex plane is shown in the top panel. The horizontal axis represents the real part of the phasor and the vertical axis is the imaginary part of the axis. The phasor itself is the blue dot. A blue line is drawn connecting the phasor to the origin to make it easier to visualize the phasor's magnitude A and the angle θ. Move the phasor to the location 0.5∠0°, which is the same thing as 0.5 + j0, or simply, the real number 0.5.

3. **Become familiar with the sinusoid panel.** The sinusoid panel is drawn rotated from the direction one would ordinarily expect. This is to help visualize the relationship between the phasor and the start point of the sinusoid – a vertical line will connect these two points when the sinusoid panel is rotated this way. The time axis is vertical, with increasing time in the downward direction, and the sinusoid's magnitude axis is horizontal, with increasing magnitude to the right. Notice how when the phasor in the top panel is real (pointing along the horizontal axis) then the sinusoid in the bottom panel looks like a cosine (starts at its maximum and decreases). Try to also imagine what the sinusoid would like like when rotated a quarter turn counter-clockwise in the more familiar orientation.

4. **Gain an intuition for phasor magnitude.** Keeping the phasor angle approximately 0°, change the magnitude to make it larger and then smaller. Notice how the magnitude of the associated sinusoid similarly becomes larger and smaller.

5. **Determine what a purely imaginary phasor represents.** You have found a purely real phasor (pointing along the horizontal axis) is associated with a cosine. Make the phasor purely negative imaginary (point it straight downwards, also described as -90° to the horizontal) with a magnitude of about 0.75. Notice that this represents a pure sine wave (starts at zero and increases to its maximum). Recall that sin(θ) = cos(θ-90°) and see that the same is true for the phasor representation.

6. **Gain an intuition for mapping a phasor to its sinusoid.** Spend a couple of minutes changing the phasor and observing the associated sinusoid. Try to predict what the sinusoid will look like for a random phasor. Mentally project the phasor straight down onto the sinusoid panel and see how it represents where the sinusoid begins (this is most obvious when the phasor is near the bottom of the panel). Discover the reason why the sinusoid panel is rotated 90°: it is because only when rotated like this can we project the phasor location down and have it represent where the sinusoid begins.

7. **Most importantly for the upcoming quiz**: Move the phasor around and try to predict what the sinusoid will look like for a given phasor.